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Oscillation of a Class of Partial Difference Equations with Unbounded Delay

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Abstract—This paper is concerned with the partial difference equation

$$A_{m+1,n} + a_{m,n}A_{m,n+1} - b_{m,n}A_{m,n} + p_{m,n}A_{\sigma(m),\tau(n)} = 0,$$

where σ and $\tau : N \to Z$ are nondecreasing, $\sigma(n)$ and $\tau(n)$ are strictly less than n for $n \in N$, $\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \tau(n) = \infty$, $\{a_{m,n}\}$, $\{b_{m,n}\}$, and $\{p_{m,n}\}$ are three real double sequences. Some oscillation criteria for this equation are obtained.

Keywords—Partial difference equation, Oscillation, Unbounded delay.

1. INTRODUCTION

The theory of difference equations, the methods used in their solutions, and their wide applications are advancing now at an exponential rate. In fact, in the last few years, several monographs and hundreds of research papers have been written, e.g., [1–9], and the references therein. On the other hand, there are also many papers that have been devoted to the development of qualitative theory of partial difference equations [10–15] which are as important as difference equations. In fact, their significance is illustrated in applications involving random walk problems, the study of molecular orbits, mathematical physics problems, and numerical difference approximation problems. To further the qualitative theory of partial difference equations, in this paper, we shall consider the partial difference equation

$$A_{m+1,n} + a_{m,n}A_{m,n+1} - b_{m,n}A_{m,n} + p_{m,n}A_{\sigma(m),\tau(n)} = 0, \tag{1}$$

where $\{a_{m,n}\}$, $\{b_{m,n}\}$, and $\{p_{m,n}\}$ are three real double sequences, $m, n = 0, 1, 2, \dots$. Let $N = \{0, 1, 2, \dots\}$ and $Z = \{\dots, -1, 0, 1, \dots\}$. For equation (1), we always assume that the following hypotheses, designated by (H), hold:

(1) σ and $\tau : N \to Z$ are nondecreasing;

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(2) $\sigma(n) < n$ and $\tau(n) < n$ for all $n \in N$;
(3) $\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \tau(n) = \infty$;
(4) $a_{m,n} \geq a$ and $b_{m,n} \leq b$, $p_{m,n} \geq 0$ for all large m and n, where a and b are two positive constants.

For example, we see that $\sigma(m) = [m/2]$ and $\tau(n) = [n/2]$ satisfy Condition (H), where $[\bullet]$ denotes the greatest integer $k \leq m/2$ and $l \leq n/2$. Hence, equation (1) includes a partial difference equation with unbounded delay. To the best of our knowledge, there are no known oscillation criteria to cover a partial difference equation with unbounded delay. We shall be interested in obtaining oscillation criteria for this equation with unbounded delay and establish some oscillation criteria for equation (1). In the same way, $\sigma(m) = m - k$ and $\tau(n) = n - l$, where k and l are two positive integers, satisfy Condition (H), and equation (1) with $\sigma(m) = m - k$ and $\tau(n) = n - l$ has been investigated in [10–12,15].

By a solution of (1), we mean a nontrivial double sequence $\{A_{m,n}\}$ satisfying (1) for $m \geq 0$ and $n \geq 0$. A solution $\{A_{m,n}\}$ of (1) is said to be eventually positive (or negative) if $A_{m,n} > 0$ (or $A_{m,n} < 0$) for all large m and n. It is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

The outline of the paper is as follows. In Section 2, we shall prove several lemmas, some of which are interesting in their own right. The oscillation theorems for equation (1) are developed in Section 3. Section 4 includes some examples to illustrate the results obtained.

2. PREPARATORY LEMMAS

To obtain our results, we need the following lemmas. The first is taken from [10].

LEMMA 1. *For $M \leq m$ and $N \leq n$, the following formal identity holds:*

$$\begin{aligned}\sum_{i=M}^{m}\sum_{j=N}^{n}\{A_{i+1,j} + aA_{i,j+1} - bA_{i,j}\} &= (1+a-b)\sum_{i=M+1}^{m}\sum_{j=N+1}^{n} A_{i,j} \\ &\quad - \sum_{j=N+1}^{n} A_{m+1,j} + (a-b)\sum_{j=N+1}^{n} A_{M,j} \qquad (2)\\ &\quad + a\sum_{i=M}^{m} A_{i,n+1} + (1-b)\sum_{i=M+1}^{m} A_{i,N} + A_{m+1,N} - bA_{M,N} \\ &= (1+a-b)\sum_{i=M+1}^{m}\sum_{j=N+1}^{n} A_{i,j} \\ &\quad + a\sum_{i=M+1}^{n} A_{i,n+1} - (a-b)\sum_{j=N+1}^{n} A_{M,j} \qquad (3)\\ &\quad + \sum_{j=N}^{n} A_{m+1,j} + (1-b)\sum_{i=M+1}^{m} A_{i,N} + aA_{M,n+1} - bA_{M,N}.\end{aligned}$$

LEMMA 2. *Assume that (H) holds and $\{A_{i,j}\}$ is an eventually positive solution of equation (1) such that $A_{i,j} > 0$ and $p_{i,j} \geq 0$ for $i \geq \sigma(M)$ and $j \geq \tau(N)$, where M and N are two sufficiently large integers. Then for any integer $k \geq 0$ and $m \geq M$ and $n \geq N$,*

$$b^{k-1} A_{m,n} \geq \sum_{i=0}^{k+1} a^i C_{k+1}^i A_{m+k+1-i,n+i}. \qquad (4)$$

PROOF. In view of (1) and (H), for $m \geq M$ and $n > N$, we have

$$bA_{m,n} \geq A_{m+1,n} + aA_{m,n+1}, \qquad (5)$$

and thus,

$$bA_{m+1,n} \geq A_{m+2,n} + aA_{m+1,n+1}, \qquad bA_{m,n+1} \geq A_{m+1,n+1} + aA_{m,n+2}.$$

Hence from (5),

$$b^2 A_{m,n} \geq A_{m+2,n} + 2aA_{m+1,n+1} + a^2 A_{m,n+2} = \sum_{i=0}^{2} a^i C_2^i A_{m+2-i,n+i}.$$

Assume that for any positive integer $k \geq 1$,

$$b^k A_{m,n} \geq \sum_{i=0}^{k} a^i C_k^i A_{m+k-i,n+i}.$$

Then for $0 \leq i \leq k$, from (5) we have

$$A_{m+k+1-i,n+i} + aA_{m+k-i,n+1+i} \leq bA_{m+k-i,n+i}.$$

From the last inequalities, we obtain

$$b^{k+1} A_{m,n} \geq \sum_{i=0}^{k} a^i C_k^i \left(A_{m+k+1-i,n+i} + aA_{m+k-i,n+1+i}\right).$$

Since

$$\begin{aligned}\sum_{i=0}^{k} a^i C_k^i \left(A_{m+k+1-i,n+i} + aA_{m+k-i,n+1+i}\right) &= A_{m+k+1,n} + \sum_{i=1}^{k} a^i C_k^i A_{m+k+1-i,n+i} \\ &\quad + \sum_{i=0}^{k+1} a^{i+1} C_k^i A_{m+k-i,n+1+i} + a^{k+1} A_{m,n+k+1} \\ &= A_{m+k+1,n} + \sum_{i=1}^{k} a^i \left(C_k^i + C_k^{i-1}\right) A_{m+k+1-i,n+i} \\ &\quad + a^{k+1} A_{m,n+k+1} = \sum_{i=0}^{k+1} a^i C_{k+1}^i A_{m+k+1-i,n+i},\end{aligned}$$

then

$$b^{k+1} A_{m,n} \geq \sum_{i=0}^{k+1} a^i C_{k+1}^i A_{m+k+1-i,n+i}.$$

The proof is completed.

COROLLARY 1. *Assume that (H) holds and $\{A_{m,n}\}$ is an eventually positive solution of equation (1) and $a \geq b$, $b \leq 1$. Then $A_{m,n}$ tends to zero as $m, n \to \infty$.*

PROOF. Assume that $A_{m,n} > 0$ and $p_{m,n} \geq 0$ for $m \geq \sigma(M)$ and $n \geq \tau(N)$, where M and N are two positive integers. By means of Lemma 2, for all positive integers k and l,

$$b^{k+l} A_{M,N} \geq a^l C_{k-l}^l A_{M+k,N+l}.$$

Thus,

$$A_{M+k,N+l} \leq \frac{b^k A_{M,N}}{C_{k+l}^l} \left(\frac{b}{a}\right)^l \to 0, \qquad \text{as } k, l \to \infty.$$

From Lemma 2, it is easy to obtain the following corollary.

COROLLARY 2. *Assume that (H) holds and $\{A_{i,j}\}$ is an eventually positive solution of equation (1) such that $A_{m,n} > 0$ and $p_{m,n} \geq 0$ for $m \geq \sigma^2(M)$ and $n \geq \tau^2(N)$; then*

$$b^{m-\sigma(m)+n-\tau(n)} A_{\sigma(m),\tau(n)} \geq a^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} A_{m,n},$$

for $m \geq M$ and $n \geq N$, where M and N are two positive integers.

LEMMA 3. *Assume that (H) holds, $a \geq b$ and $b \leq 1$, and $\{A_{i,j}\}$ is an eventually positive solution of equation (1). If there exists $B > 0$ such that for sufficiently large M and N,*

$$\sum_{i=M}^{m} \sum_{j=N}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \geq B, \tag{6}$$

then

$$bA_{M,N} \geq A_{m+1,N} + BA_{\sigma(m),\tau(n)} \tag{7}$$

and

$$bA_{M,N} \geq aA_{M,n+1} + BA_{\sigma(m),\tau(n)}. \tag{8}$$

PROOF. In view of (1),(2) and Lemma 2, for sufficiently large M and N, we obtain

$$\begin{aligned}
0 &\geq \sum_{i=M}^{m} \sum_{j=N}^{n} \left(A_{i+1,j} + aA_{i,j+1} - bA_{i,j} + p_{i,j} A_{\sigma(i),\tau(j)}\right) \\
&\geq \sum_{i=M}^{m} \sum_{j=N}^{n} p_{i,j} A_{\sigma(i),\tau(j)} + A_{m+1,N} - bA_{M,N} \\
&\geq \sum_{i=M}^{m} \sum_{j=N}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} A_{\sigma(m),\tau(n)} + A_{m+1,N} - bA_{M,N} \\
&\geq BA_{\sigma(m),\tau(n)} + A_{m+1,N} - bA_{M,N},
\end{aligned}$$

and hence, inequality (7) holds. On the other hand, from (1) and (3) we find

$$0 \geq \sum_{i=M}^{m} \sum_{j=N}^{n} p_{i,j} A_{\sigma(i),\tau(j)} + aA_{M,n+1} - bA_{M,N}.$$

Following a similar argument as above, we obtain (8). The proof is completed.

LEMMA 4. *Assume that (H) holds, $a \geq b$ and $b \leq 1$, and $\{A_{i,j}\}$ is an eventually positive solution of equation (1), and for all large m and n,*

$$\sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \geq B > 0. \tag{9}$$

Then for all large m and n,

$$\frac{A_{\sigma^2(m),\tau(n)}}{A_{\sigma(m),n}} \leq \left(\frac{2b}{B}\right)^4,$$

where $\sigma^0(m) = m$ and $\sigma^k(m) = \sigma(\sigma^{k-1}(m))$, $k = 1, 2, \dots$.

PROOF. By means of (9), for all large m and n, there exists an integer $\overline{m}$ such that $m \in \{\sigma(\overline{m}), \sigma(\overline{m}) + 1, \dots, \overline{m}\}$ and

$$\sum_{i=\sigma(\overline{m})}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \geq \frac{B}{2},$$

and

$$\sum_{i=m}^{\overline{m}} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \geq \frac{B}{2}.$$

By means of Lemma 3, we have

$$bA_{\sigma(m),\tau(n)} \geq A_{m+1,\tau(n)} + \frac{B}{2} A_{\sigma(m),\tau(n)},$$
$$bA_{m,\tau(n)} \geq A_{\overline{m}+1,\tau(n)} + \frac{B}{2} A_{\sigma(\overline{m}),\tau(n)}.$$

Hence, $A_{m,\tau(n)} \geq (B/2b)^2 A_{\sigma(m),\tau(n)}$ for all large m and n. Similarly, $A_{\sigma(m),n} \geq (B/2b)^2 A_{\sigma(m),\tau(n)}$ for all large m and n. Thus, for all large m and n,

$$\frac{A_{\sigma^2(m),\tau(n)}}{A_{\sigma(m),n}} = \frac{A_{\sigma^2(m),\tau(n)}}{A_{\sigma(m),\tau(n)}} \cdot \frac{A_{\sigma(m),\tau(n)}}{A_{\sigma(m),n}} \leq \left(\frac{2b}{B}\right)^4.$$

The proof is completed.

3. MAIN RESULTS

THEOREM 1. *Assume that (H) holds and*

$$\limsup_{m,n\to\infty} p_{m,n} \frac{1}{b^{m-\sigma(m)+1}} \left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} > 1; \tag{10}$$

then every solution of equation (1) is oscillatory.

PROOF. Suppose to the contrary, there is an eventually positive solution $\{A_{m,n}\}$ of (1) such that $A_{m,n} > 0$ and $p_{m,n} \geq 0$ for $m \geq \sigma^2(M)$ and $n \geq \tau^2(N)$, where M and N are positive integers. By means of Corollary 2, we have for $m \geq M$ and $n > N$,

$$A_{m+1,n} + aA_{m,n+1} - bA_{m,n} + p_{m,n} \frac{1}{b^{m-\sigma(m)}} \left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} A_{m,n} \leq 0,$$

i.e.,

$$p_{m,n} \frac{1}{b^{m-\sigma(m)+1}} \left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} \leq 1, \qquad \text{for } m \geq M \quad \text{and} \quad n \geq N,$$

which is a contradiction to (10). The proof is completed.

The following two corollaries can easily be derived from Theorem 1, and their proofs are thus omitted.

COROLLARY 3. *Assume that (H) holds, $a > b$, and $b \leq 1$. If either $\lim_{m\to\infty}(m - \sigma(m)) = \infty$ or $\lim_{n\to\infty}(n - \tau(n)) = \infty$ hold, and $\limsup_{m,n\to\infty} p_{m,n} > 0$, then every solution of equation (1) is oscillatory.*

COROLLARY 4. *Assume that (H) holds, $\sigma(m) = m - \sigma$, and $\tau(n) = n - \tau$, where σ and τ are two positive integers. If*

$$\limsup_{m,n\to\infty} p_{m,n} > \frac{b^{\sigma+1}}{C_{\sigma+\tau}^{\tau}} \left(\frac{b}{a}\right)^{\tau},$$

then every solution of equation (1) oscillates.

If (10) does not hold, then we have the following result.

THEOREM 2. *Assume that (H) holds and*

$$\limsup_{m,n\to\infty} \frac{1}{d_{m,n}} \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C^{\tau(n)-\tau(j)}_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)} > 1; \tag{11}$$

then every solution of equation (1) is oscillatory, where

$$\begin{aligned} d_{m,n} &= b, && a \ge b, \quad b \le 1, \\ &= b^{m-\sigma(m)+1}, && a \ge b, \quad b \ge 1, \\ &= b\left(\frac{b}{a}\right)^{n-\tau(n)}, && a \le b, \quad b \le 1, \\ &= b\left[\left(\frac{b}{a}\right)^{n-\tau(n)} - 1 + b^{m-\sigma(m)}\right], && a \le b, \quad b \ge 1, \quad b-a \le 1, \\ &= b\left[\left(\frac{b}{a}\right)^{n-\tau(n)} - 1 + b^{m-\sigma(m)}\right] - (1+a-b) \sum_{i=\sigma(m)+1}^{m} \sum_{j=\tau(n)+1}^{n} \frac{b^{i-\sigma(m)}}{C^{j-\tau(n)}_{i-\sigma(m)+j-\tau(n)}} \left(\frac{b}{a}\right)^{j-\tau(n)}, \\ & && a \le b, \quad b \ge 1, \quad b-a \ge 1. \end{aligned}$$

PROOF. Assume that there exists an eventually positive solution $\{A_{m,n}\}$ of equation (1) such that $A_{m,n} > 0$ and $p_{m,n} \ge 0$ for $m \ge \sigma^2(M)$ and $n \ge \tau^2(N)$, where M and N are two sufficiently large positive integers. Then in view of (1), Lemma 1, and Lemma 2, for $m \ge M$ and $n \ge N$, we have

$$A_{m+1,n} + aA_{m,n+1} - bA_{m,n} + p_{m,n}A_{\sigma(m),\tau(n)} \le 0,$$

and thus,

$$\begin{aligned} 0 &\ge \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} \left[A_{i+1,j} + aA_{i,j+1} - bA_{i,j} + p_{i,j}A_{\sigma(i),\tau(j)}\right] \\ &= \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j}A_{\sigma(i),\tau(j)} + (1+a-b) \sum_{i=\sigma(m)+1}^{m} \sum_{j=\tau(n)+1}^{n} A_{i,j} + \sum_{j=\tau(n)+1}^{n} A_{m+1,j} \\ &\quad + a \sum_{i=\sigma(m)}^{m} A_{i,n+1} + (a-b) \sum_{j=\tau(n)+1}^{n} A_{\sigma(m),j} \\ &\quad + (1-b) \sum_{i=\sigma(m)+1}^{m} A_{i,\tau(n)} + A_{m+1,\tau(n)} - bA_{\sigma(m),\tau(n)}. \end{aligned} \tag{12}$$

CASE A. $a \ge b$, $b \le 1$.

Inequality (12) provides

$$0 \ge \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j}A_{\sigma(i),\tau(j)} - bA_{\sigma(m),\tau(n)}.$$

By Lemma 2, we have for $\sigma(m) \le i \le m$ and $\tau(n) \le j \le n$,

$$b^{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)} A_{\sigma(i),\tau(j)} \ge a^{\tau(n)-\tau(j)} C^{\tau(n)-\tau(j)}_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)} A_{\sigma(m),\tau(n)}.$$

It follows that

$$0 \ge \left\{ \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C^{\tau(n)-\tau(j)}_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)} - b \right\} A_{\sigma(m),\tau(n)},$$

or equivalently,

$$\sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \leq b,$$

which is a contradiction to (11).

CASE B. $a \geq b$, $b \geq 1$.

It follows from (12) and Lemma 2 that

$$\begin{aligned} 0 &\geq \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} A_{\sigma(i),\tau(j)} + (1-b) \sum_{i=\sigma(m)+1}^{m} A_{i,\tau(n)} - bA_{\sigma(m),\tau(n)} \\ &\geq \left\{ \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \right. \\ &\qquad \left. + (1-b) \sum_{i=\sigma(m)+1}^{m} b^{i-\sigma(m)} - b \right\} A_{\sigma(m),\tau(n)}. \end{aligned}$$

A similar contradiction as in Case A is thus obtained.

CASE C. $a \leq b$, $b \leq 1$.

Since $b \leq 1$, we have $1 + a - b > 0$. Consequently, from (12) and Lemma 2, we find

$$\begin{aligned} 0 &\geq \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} A_{\sigma(i),\tau(j)} + (a-b) \sum_{j=\tau(n)+1}^{n} A_{\sigma(m),j} - bA_{\sigma(m),\tau(n)} \\ &\geq \left\{ \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \right. \\ &\qquad \left. + (a-b) \sum_{j=\tau(n)+1}^{n} \left(\frac{b}{a}\right)^{j-\tau(n)} - b \right\} A_{\sigma(m),\tau(n)}, \end{aligned}$$

which is a contradiction to (11).

CASE D. $a \leq b$, $b \geq 1$, $b - a \leq 1$.

Since $b - a \leq 1$, then $1 + a - b \geq 0$. Hence, it follows from (12) that

$$\begin{aligned} 0 &\geq \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} A_{\sigma(i),\tau(j)} + (a-b) \sum_{j=\tau(n)+1}^{n} A_{\sigma(m),j} + (1-b) \sum_{i=\sigma(m)+1}^{m} A_{i,\tau(n)} - bA_{\sigma(m),\tau(n)} \\ &\geq \left\{ \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \right. \\ &\qquad \left. + (a-b) \sum_{j=\tau(n)+1}^{n} \left(\frac{b}{a}\right)^{j-\tau(n)} + (1-b) \sum_{i=\sigma(m)+1}^{m} b^{i-\sigma(m)} - b \right\} A_{\sigma(m),\tau(n)}. \end{aligned}$$

The rest of the proof is similar to that of Cases A–C.

CASE E. $a \leq b$, $b \geq 1$, $b - a \geq 1$.

Since $b-a\ge 1$, then $1+a-b\le 0$. Therefore, from (12) and Lemma 2, we find

$$\begin{aligned}
0 \ge & \sum_{i=\sigma(m)}^{m}\sum_{j=\tau(n)}^{n} p_{i,j}A_{\sigma(i),\tau(j)} + (1+a-b)\sum_{i=\sigma(m)+1}^{m}\sum_{j=\tau(n)+1}^{n} A_{i,j} \\
& + (a-b)\sum_{j=\tau(n)+1}^{n} A_{\sigma(m),j} + (1-b)\sum_{i=\sigma(m)+1}^{m} A_{i,\tau(n)} - bA_{\sigma(m),\tau(n)} \\
\ge & \left\{ \sum_{i=\sigma(m)}^{m}\sum_{j=\tau(n)}^{n} p_{i,j}\frac{1}{b^{\sigma(m)-\sigma(i)}}\left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \right. \\
& + (a-b)\sum_{j=\tau(n)+1}^{n}\left(\frac{b}{a}\right)^{j-\tau(n)} + (1+a-b)\sum_{i=\sigma(m)+1}^{m}\sum_{j=\tau(n)+1}^{n}\frac{b^{i-\sigma(m)}}{C_{i-\sigma(m)+j-\tau(n)}^{j-\tau(n)}}\left(\frac{b}{a}\right)^{j-\tau(n)} \\
& \left. + (1-b)\sum_{i=\sigma(m)+1}^{m} b^{i-\sigma(m)} - b \right\} A_{\sigma(m),\tau(n)},
\end{aligned}$$

which leads to the required contradiction. The proof is completed.

Noting that if $\sigma(m)=m-\sigma$ and $\tau(n)=n-\tau$, then $d_{m,n}(=d)$ is a constant. Thus, from Theorem 2, we can obtain the following corollary.

COROLLARY 5. *Assume that (H) holds, $\sigma(m)=m-\sigma$, and $\tau(n)=n-\tau$, where σ and τ are two positive integers. If*

$$\limsup_{m,n\to\infty}\sum_{i=m-\sigma}^{m}\sum_{j=n-\tau}^{n} p_{i,j}\frac{1}{b^{m-i}}\left(\frac{a}{b}\right)^{n-j} C_{m-i+n-j}^{n-j} > d,$$

then every solution of equation (1) oscillates.

In view of (H), $a\ge b$, and $b\le 1$, we have

$$\frac{1}{b^{\sigma(m)-\sigma(i)}}\left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \ge 1, \qquad \text{for all } i\le m \text{ and } j\le n.$$

Hence, from Theorem 2, it is easy to obtain the next corollary.

COROLLARY 6. *Assume that (H) holds, $a\ge b$, and $b\le 1$. If*

$$\limsup_{m,n\to\infty}\sum_{i=\sigma(m)}^{m}\sum_{j=\tau(n)}^{n} p_{i,j} > b,$$

then every solution of equation (1) is oscillatory.

If (10) and (11) do not hold, then we have the following results.

THEOREM 3. *Assume that (H) holds, $a\ge b$, and $b\le 1$. If*

$$\lim_{m,n\to\infty}\frac{1}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} = \infty, \tag{13}$$

$$\liminf_{m,n\to\infty}\sum_{i=\sigma(m)}^{m-1}\sum_{j=\tau(n)}^{n-1} p_{i,j}\frac{1}{b^{\sigma(m)-\sigma(i)}}\left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} > 0, \tag{14}$$

then every solution of (1) oscillates.

PROOF. In view of (13) and Corollary 2, we obtain

$$\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} \ge \frac{1}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} \to \infty, \qquad \text{as } m\to\infty \text{ and } n\to\infty.$$

On the other hand, from (14) and Lemma 4, we have $\limsup_{m,n\to\infty} A_{\sigma(m),\tau(n)}/A_{m,n}$ exists, which is a contradiction. The proof is completed.

COROLLARY 7. *Assume that (H) holds, $a \geq b$, and $b \leq 1$. If either $\lim_{m\to\infty}(m-\sigma(m)) = \infty$ or $\lim_{n\to\infty}(n-\tau(n)) = \infty$ holds, and*

$$\liminf_{m,n\to\infty} \sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} > 0,$$

then every solution of equation (1) is oscillatory.

Noting that if $a \geq b$ and $b \leq 1$ and either $\lim_{m\to\infty}(m-\sigma(m)) = \infty$ or $\lim_{n\to\infty}(n-\tau(n)) = \infty$, then (13) holds, and it is easy to see that (14) holds. Therefore, Corollary 7 holds.

Let

$$\lambda = \lambda_{m,n} = \frac{2(m-\sigma(m))(n-\tau(n))}{m-\sigma(m)+n-\tau(n)}, \tag{15}$$

$$\theta = \theta_{m,n} = \sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} \frac{1}{b^{i-\sigma(i)}} \left(\frac{a}{b}\right)^{j-\tau(j)} C_{i-\sigma(i)+j-\tau(j)}^{j-\tau(j)}, \tag{16}$$

$$k = k_{m,n} = \frac{(m-\sigma(m))^2}{m-\sigma(m)+n-\tau(n)}, \tag{17}$$

$$l = l_{m,n} = \frac{(n-\tau(n))^2}{m-\sigma(m)+n-\tau(n)}, \tag{18}$$

$$s = s_{m,n} = \frac{2^{\lambda_{m,n}}(1+\lambda_{m,n})^{1+\lambda_{m,n}}}{\lambda_{m,n}^{\lambda_{m,n}}(m-\sigma(m))(n-\tau(n))C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)}}. \tag{19}$$

THEOREM 4. *Assume that (H) holds. If $\liminf_{m,n\to\infty}\{b_{m,n}/p_{m,n}\} < \infty$ and*

$$\liminf_{m,n\to\infty} \frac{1}{b^{m-\sigma(m)}} \left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} > 0, \tag{20}$$

$$\liminf_{m,n\to\infty} s_{m,n}\theta_{m,n} \frac{1}{b^{1+k_{m,n}}} \left(\frac{a}{b}\right)^{l_{m,n}} = \liminf_{m,n\to\infty} s\theta \frac{1}{b^{1+k}} \left(\frac{a}{b}\right)^{l} > 1, \tag{21}$$

then every solution of (1) oscillates.

PROOF. Suppose to the contrary, there exists an eventually positive solution $\{A_{m,n}\}$ of (1) such that $A_{m,n} > 0$ and $p_{m,n} \geq 0$ for $m \geq \sigma^3(M)$ and $n \geq \tau^3(N)$, where M and N are two positive integers. From (1), we have for $m \geq M$ and $n \geq N$,

$$\frac{2\sqrt{a}}{b} \cdot \frac{(A_{m+1,n}A_{m,n+1})^{1/2}}{A_{m,n}} \leq \frac{A_{m+1,n} + aA_{m,n+1}}{bA_{m,n}} = 1 - p_{m,n}\frac{A_{\sigma(m),\tau(n)}}{bA_{m,n}}.$$

Hence, by means of Corollary 2 and the well-known inequality between the arithmetic and geometric mean, we obtain

$$\begin{aligned}
&\left(\frac{2\sqrt{a}}{b}\right)^{(m-\sigma(m))(n-\tau(n))} \prod_{i=\sigma(m)}^{m-1} \prod_{j=\tau(n)}^{n-1} \frac{(A_{i+1,j}A_{i,j+1})^{1/2}}{A_{i,j}} \\
&\leq \prod_{i=\sigma(m)}^{m-1} \prod_{j=\tau(n)}^{n-1} \left(1 - p_{i,j}\frac{A_{\sigma(i),\tau(j)}}{bA_{i,j}}\right) \\
&\leq \left(1 - \frac{1}{b(m-\sigma(m))(n-\tau(n))} \sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j}\frac{A_{\sigma(i),\tau(j)}}{A_{i,j}}\right)^{(m-\sigma(m))(n-\tau(n))} \qquad (22) \\
&\leq \left(1 - \frac{1}{b(m-\sigma(m))(n-\tau(n))} \sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j}\frac{1}{b^{i-\sigma(i)}}\left(\frac{a}{b}\right)^{j-\tau(j)}\right.
\end{aligned}$$

$$\times\, C_{i-\sigma(i)-j-\tau(j)}^{j-\tau(j)}\Bigg)^{(m-\sigma(m))(n-\tau(n))}$$

$$= \left(1 - \frac{\theta_{m,n}}{b(m-\sigma(m))(n-\tau(n))}\right)^{(m-\sigma(m))(n-\tau(n))}. \tag{22(cont.)}$$

Since

$$\begin{aligned}
\prod_{i=\sigma(m)}^{m-1}\prod_{j=\tau(n)}^{n-1}\frac{(A_{i+1,j}A_{i,j+1})^{1/2}}{A_{i,j}} &= \prod_{i=\sigma(m)}^{m-1}\left(\frac{A_{i,n}}{A_{i,\tau(n)}}\right)^{1/2}\prod_{j=\tau(n)}^{n-1}\left(\frac{A_{m,j}}{A_{\sigma(m),j}}\right)^{1/2}\\
&\geq \left(\frac{A_{m,n}}{A_{\sigma(m),\tau(n)}}\right)^{(m-\sigma(m)+n-\tau(n))/2}\prod_{i=\sigma(m)}^{m-1}\frac{1}{b^{m-i}}\cdot\frac{1}{b^{i-\sigma(m)}}\\
&\quad\times\prod_{j=\tau(n)}^{n-1}\left(\frac{a}{b}\right)^{n-j}\cdot\left(\frac{a}{b}\right)^{j-\tau(n)}\\
&= \frac{a^{(n-\tau(n))^2}}{b^{(m-\sigma(m))^2+(n-\tau(n))^2}}\left(\frac{A_{m,n}}{A_{\sigma(m),\tau(n)}}\right)^{(m-\sigma(m)+n-\tau(n))/2},
\end{aligned}$$

then from (22) and the inequality $x(1-x)^\lambda \leq \lambda^\lambda/(1+\lambda)^{1+\lambda}$ for $x \in (0,1)$, we have

$$\begin{aligned}
\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} &\geq \frac{(2\sqrt{a}/b)^\lambda\,(1/b^{2k})\,(a/b)^{2l}}{(1-\theta/[b(m-\sigma(m))(n-\tau(n))])^\lambda}\\
&\geq \left(\frac{2\sqrt{a}}{b}\right)^\lambda\frac{1}{b^{2k}}\left(\frac{a}{b}\right)^{2l}\frac{\theta}{b(m-\sigma(m))(n-\tau(n))}\frac{(1+\lambda)^{1+\lambda}}{\lambda^\lambda}\\
&= s\theta\frac{1}{b^{k+1}}\left(\frac{a}{b}\right)^{l}\frac{1}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)}C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)},
\end{aligned}\tag{23}$$

where λ, θ, l, k, and s are defined by (15)–(19). In view of (21), there is a constant $r > 1$ such that

$$s\theta\frac{1}{b^{k+1}}\left(\frac{a}{b}\right)^{l} > r, \qquad \text{for all large } m \text{ and } n.$$

Hence from (23), we have

$$A_{\sigma(m),\tau(n)} \geq \frac{r}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)}C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)}A_{m,n}. \tag{24}$$

Substituting (24) into (1), we get for all large m and n,

$$\begin{aligned}
\frac{2\sqrt{a}}{b}\cdot\frac{(A_{m+1,n}A_{m,n+1})^{1/2}}{A_{m,n}} &\leq \frac{A_{m+1,n}+aA_{m,n+1}}{bA_{m,n}} \leq 1 - p_{m,n}\frac{A_{\sigma(m),\tau(n)}}{bA_{m,n}}\\
&\leq 1 - \frac{r}{b}\cdot p_{m,n}\frac{1}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)}C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)}.
\end{aligned}$$

Hence, for all large m and n,

$$\begin{aligned}
&\left(\frac{2\sqrt{a}}{b}\right)^{(m-\sigma(m))(n-\tau(n))}\prod_{i=\sigma(m)}^{m-1}\prod_{j=\tau(n)}^{n-1}\frac{(A_{i+1,j}A_{i,j+1})^{1/2}}{A_{i,j}}\\
&\qquad\leq \prod_{i=\sigma(m)}^{m-1}\prod_{j=\tau(n)}^{n-1}\left(1-\frac{r}{b}\cdot p_{i,j}\frac{1}{b^{i-\sigma(i)}}\left(\frac{a}{b}\right)^{j-\tau(n)}C_{i-\sigma(i)+j-\tau(n)}^{j-\tau(n)}\right).
\end{aligned}$$

Thus, as in the above proof, for all large m and n we can obtain

$$\begin{aligned}\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} &\geq \frac{\left(2\sqrt{a}/b\right)^{\lambda}\left(1/b^{2k}\right)(a/b)^{2l}}{\left(1 - r\theta/[b(m-\sigma(m))(n-\tau(n))]\right)^{\lambda}} \\ &\geq rs\theta \frac{1}{b^{k+1}}\left(\frac{a}{b}\right)^{l}\frac{1}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} \\ &\geq \frac{r^2}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)}.\end{aligned}$$

By induction, we get for any positive integer N,

$$\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} \geq \frac{r^N}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)}, \qquad \text{for all large } m \text{ and } n.$$

In view of (20), we have

$$\lim_{m,n\to\infty} \frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} = +\infty. \tag{25}$$

On the other hand, in view of (1), we have

$$\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} \leq \frac{b_{m,n}}{p_{m,n}}, \qquad \text{for all large } m \text{ and } n.$$

Since $\liminf_{m,n\to\infty} b_{m,n}/p_{m,n} < \infty$, then

$$\liminf_{m,n\to\infty} \frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} < +\infty,$$

which is a contradiction to (25). The proof is completed.

COROLLARY 8. *Assume that (H) holds, $\sigma(m) = m - \sigma$, and $\tau(n) = n - \tau$, where σ and τ are two positive integers. If $a \geq b$, $b \leq 1$, and*

$$\liminf_{m,n\to\infty} \frac{1}{kl} \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} p_{i,j} > \frac{\lambda^{\lambda} b^{1+k+\sigma}}{2^{\lambda}(1+\lambda)^{1+\lambda}} \cdot \left(\frac{b}{a}\right)^{l+\tau},$$

then every solution of equation (1) is oscillatory, where $\lambda = 2\sigma\tau/(\sigma+\tau)$, and k, l are defined by (17) and (18).

REMARK. It is easy to see that Theorem 3.4 in [12] is a special case of Corollary 8, and so is Corollary 3.2 in [15].

4. EXAMPLES

In this section, we will illustrate the use of our oscillation criteria for partial difference equations with delay by means of some examples. First, we illustrate the differences between our results and that of Wong and Agarwal [10].

EXAMPLE 1. Consider the partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n} A_{m+1,n+1} = 0, \tag{26}$$

where

$$p_{m,n} = \begin{cases} 0.6, & m = n = 16^k, \\ 0, & \text{otherwise}, \end{cases} \qquad k = 0, 1, 2, \dots.$$

It is easy to see that (H) holds and $\limsup_{m,n\to\infty} p_{m,n} = 0.6 > 0.5 = 1/C_2^1$. Thus, by means of Corollary 4, every solution of (26) oscillates, but the same conclusion cannot be inferred from Theorem 3.1 in [10].

Examples 2 and 3 illustrate that our results can be used to judge oscillation for partial difference equations with unbounded delay. To the best of our knowledge, there is no known oscillation criterion for these equations.

EXAMPLE 2. Consider the partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{m-2,[n/2]} = 0, \tag{27}$$

where

$$p_{m,n} = \begin{cases} \dfrac{1}{n+1}, & 2^{3k-1} \le n < 2^{3k} \text{ and } m \ge 0, \\ 0, & \text{otherwise}, \end{cases} \qquad k = 1, 2, \dots.$$

It is easy to see that (H) holds and if $n = 2^{3k-1}$, then

$$\sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} = \sum_{i=m-2}^{m} \sum_{j=2^{3k-1}}^{2^{3k}-1} \frac{1}{j+1} \ge \frac{3 \cdot 2^{3k-1}}{2^{3k}} = \frac{3}{2}.$$

Hence, $\limsup_{m,n\to\infty} \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \ge 3/2 > 1$. Thus, by means of Corollary 5, every solution of (27) oscillates.

EXAMPLE 3. Consider the equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{[m/2],[n/2]} = 0, \tag{28}$$

where $p_{m,n} = 1/(m+1)(n+1)$, $m, n = 0, 1, 2, \dots$. We can see that if $m = 2k$, $n = 2l$, $k, l = 1, 2, \dots$, then

$$\sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} = \sum_{i=k}^{2k-1} \sum_{j=l}^{2l-1} \frac{1}{(i+1)(j+1)} \ge \sum_{i=k}^{2k-1} \frac{1}{i+1} \cdot \frac{l}{2l} > \frac{1}{2} \cdot \frac{k}{2k} = \frac{1}{4},$$

and in same method, if $m = 2k$ and $n = 2l-1$ or $m = 2k-1$ and $n = 2l$ or $m = 2k-1$ and $n = 2l-1$, $k, l = 1, 2, \dots$, then

$$\sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} \ge \frac{1}{4}.$$

Hence, $\liminf_{m,n\to\infty} \sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} \ge 1/4 > 0$. Thus, by means of Corollary 7, every solution of (28) oscillates.

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